INSTABILITY OF WORMHOLES WITH A NONMINIMALLY COUPLED SCALAR FIELD

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Static, spherically symmetric, traversable wormholes, induced by massless, nonminimally coupled scalar fields in general relativity, are shown to be unstable under spherically symmetric perturbations. The instability is related to blowing-up of the effective gravitational constant on a certain sphere.

1. Introduction

As is widely known, traversable wormholes as solutions to the Einstein equations can only exist with exotic matter, more precisely, if the energy-momentum tensor (EMT) of the matter source of gravity violates the local and averaged null energy condition (NEC) $T_{\mu\nu}k^\mu k^\nu \geq 0$, $k_\mu k^\mu = 0$ [1]. Scalar fields provide good examples of such matter: on the one hand, in many particular models they do exhibit exotic properties, on the other, many exact solutions are known for gravity with scalar sources.

Consider, for instance, the general class of scalar-tensor theories (STT), where gravity is characterized by the metric $g_{\mu\nu}$ and the scalar field ϕ; the action is

$$S = \int d^4x \sqrt{g} \{ f(\phi)\mathcal{R}[g] + h(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \\ - 2U(\phi) + 16\pi G\, L_m \}. \quad (1)$$

Here $\mathcal{R}[g]$ is the scalar curvature, f, h and U are certain functions of ϕ, varying from theory to theory, L_m is the matter Lagrangian, and G is the gravitational constant, not necessarily coinciding with its Newtonian value. Exact static, spherically symmetric solutions are known, in particular, for the case of massless scalar-vacuum fields ($U = 0$, $L_m = 0$) [2, 3]. Wormholes form one of the generic classes of solutions in theories where the kinetic term in (1) is negative [3].

The energy conditions, NEC in particular, are, however, violated as well by "less exotic" sources, such as the so-called nonminimally coupled scalar fields in general relativity, represented by the action (1) with the functions

$$f(\phi) = 1 - \xi\phi^2, \quad \xi = \text{const}; \qquad h(\phi) \equiv 1. \quad (2)$$

It turns out that, with such a field, there exist static, spherically symmetric wormhole solutions, as shown in Ref. [3] (and recently discussed in Ref. [5]) for conformal coupling, $\xi = 1/6$, and in Ref. [6] for any $\xi > 0$. The

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easiness of violating the energy conditions, becoming so evident due to the appearance of wormhole solutions, even made Barceló and Visser discuss a "restricted domain of application of the energy conditions" [6].

We show in this paper that the scalar-vacuum wormhole solutions, previously obtained for the theory (1), (2), are unstable under spherically symmetric perturbations. The instability turns out to be of catastrophic nature: the increment of perturbation growth has no upper bound, hence, within linear perturbation theory, such a wormhole, if once formed, should decay immediately and instantaneously. A full dynamical solution (yet to be found) would probably show a finite but still enormous decay rate.

A more general observation is that, although the energy conditions are more or less easily violated, it is a much more arduous task to create a viable wormhole.

2. Wormhole solutions with nonminimally coupled scalar fields

2.1. The solution

The general STT action (1) is simplified by the well-known conformal mapping [4]

$$g_{\mu\nu} = \overline{g}_{\mu\nu}/|f(\phi)|, \quad (3)$$

accompanied by the scalar field transformation $\phi \mapsto \psi$ such that

$$\frac{d\psi}{d\phi} = \pm \frac{\sqrt{|l(\phi)|}}{f(\phi)}, \qquad l(\phi) \stackrel{\text{def}}{=} fh + \frac{3}{2}\left(\frac{df}{d\phi}\right)^2. \quad (4)$$

In terms of $\overline{g}_{\mu\nu}$ and ψ the action takes the form

$$S = \int d^4x \sqrt{\overline{g}}(\text{sign}\, f)\{\mathcal{R}[\overline{g}] + [\text{sign}\, l(\phi)]\overline{g}^{\mu\nu}\psi_{,\mu}\psi_{,\nu}\} \quad (5)$$

(for $U = L_m = 0$, up to a boundary term which does not affect the field equations). Here $R[\overline{g}]$ is the Ricci scalar obtained from $\overline{g}_{\mu\nu}$.

The space-time $\mathbb{M}[g]$ with the metric $g_{\mu\nu}$ is referred to as the *Jordan conformal frame*, generally regarded to

be the physical frame in STT; the *Einstein conformal frame* $\overline{\mathbb{M}}[\overline{g}]$ with the field ψ then plays an auxiliary role. The action (5) corresponds to conventional general relativity if $f > 0$, and the normal sign of scalar kinetic energy is obtained for $l(\phi) > 0$.

The general static, spherically symmetric solution to the Einstein-scalar equations that follow from (5), was first found by Fisher [2] and was repeatedly rediscovered afterwards. Let us write it in the form suggested in [3], restricting ourselves to the "normal" case $l > 0$:

$$\psi(u) = Cu + \psi_0, \qquad (6)$$

$$ds_{\mathrm{E}}^2 = \mathrm{e}^{2\gamma(u)}dt^2 - \mathrm{e}^{2\alpha(u)}du^2 - \mathrm{e}^{2\beta(u)}d\Omega^2$$

$$= \mathrm{e}^{-2hu}dt^2 - \frac{k^2\,\mathrm{e}^{2hu}}{\sinh^2(ku)}\left[\frac{k^2du^2}{\sinh^2(ku)} + d\Omega^2\right] \qquad (7)$$

where the subscript "E" stands for the Einstein frame; $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the linear element on a unit sphere; C (the scalar charge), $h > 0$ (the mass in geometric units), $k > 0$ and ψ_0 are integration constants, of which the first three are related by

$$k^2 = h^2 + \tfrac{1}{2}C^2. \qquad (8)$$

Without loss of generality we put $C > 0$ and $\psi_0 = 0$.

In this solution we are using the harmonic radial coordinate[3] $u \in \mathbb{R}_+$ in $\overline{\mathbb{M}}[\overline{g}]$, satisfying the coordinate condition $\alpha = 2\beta + \gamma$. The value $u = 0$ corresponds to flat spatial infinity, whereas $u \to \infty$ is a naked singularity, situated at the centre of the system (i.e., $\overline{g}_{\theta\theta} = \mathrm{e}^{2\beta} \to 0$), with an infinite value of the scalar ψ.

All the corresponding Jordan-frame solutions for $l(\phi) > 0$ are obtained from (6), (7) using (3), (4).

Let us now turn to wormhole solutions for the non-minimal coupling (2), $\xi > 0$. The transformation (4) takes the form

$$\frac{d\psi}{d\phi} = \frac{\sqrt{|1 - \phi^2(\xi - 6\xi^2)|}}{1 - \xi\phi^2}, \qquad (9)$$

where, without loss of generality, we choose the plus sign before the square root. We assume that spatial infinity in the Jordan space-time $\mathbb{M}$ corresponds to $|\phi| < 1/\sqrt{\xi}$, where $f(\phi) > 0$, so that the gravitational coupling has its normal sign.

Wormholes are obtained in $\mathbb{M}[g]$ when the solution is smoothly continued in $\mathbb{M}[g]$ through the sphere $\mathbb{S}_{\mathrm{trans}}$ ($u = \infty$, $\phi = 1/\sqrt{\xi}$ which is singular in $\overline{\mathbb{M}}[\overline{g}]$). The infinity of the conformal factor $1/f$ thus compensates the zero of both $\overline{g}_{tt}$ and $\overline{g}_{\theta\theta}$ simultaneously. This happens when, in accord with (8),

$$k = 2h = 2C/\sqrt{6}, \qquad (10)$$

[3]Another coordinate, r, used, in particular, in Refs. [5, 6], corresponding to the coordinate gauge $\alpha + \gamma = 0$, in terms of u is $r = 2k/(1 - \mathrm{e}^{-2ku})$, the metric in terms of r has the form

$$ds_{\mathrm{E}}^2 = (1 - 2\eta/r)^a dt^2 - (1 - 2\eta/r)^{-a}\left[dr^2 + r^2(1 - 2\eta/r)d\Omega^2\right],$$

and the constants are related by $\eta = k$, $a = h/k$.

which selects a special subfamily among all solutions. We will restrict the consideration to this subfamily.

Eq. (9) shows that $\psi \to \infty$ as $\phi \to 1/\sqrt{\xi} - 0$. Therefore the solution in $\mathbb{M}[g]$ has been built so far for the region where $\phi < 1/\sqrt{\xi}$. Quite a similar solution exists, however, for $\phi > 1/\sqrt{\xi}$, since the Einstein-frame equations due to (5) do not change when f changes its sign. The metric $\overline{g}_{\mu\nu}$ of this second Einstein-frame manifold[4] $\overline{\mathbb{M}}'$ should also be regularized by the factor $1/f$ on $\mathbb{S}_{\mathrm{trans}}$, hence the integration constants in it should satisfy the condition (10). Moreover, one can verify that to provide a smooth transition in the Jordan-frame metric $g_{\mu\nu}$ through $\mathbb{S}_{\mathrm{trans}}$, all the constants k, h, C and ψ_0 should coincide in $\overline{\mathbb{M}}$ and $\overline{\mathbb{M}}'$.

The latter statement can only be proved using a coordinate which is common on both sides of $\mathbb{S}_{\mathrm{trans}}$, hence a coordinate other than u. It can be seen that the whole space-time $M[g]$ can be described in terms of the coordinate ϕ.

2.2. The scalar field ϕ as a coordinate

Eq. (9) is integrated giving [6] [5]

$$\psi = -\sqrt{3/2}\ln[B(\phi)H^2(\phi)] \qquad (11)$$

where

$$B(\phi) = B_0\frac{\sqrt{1 - \xi(1 - 6\xi)\phi^2} - \sqrt{6}\xi\phi}{\sqrt{1 - \xi(1 - 6\xi)\phi^2} + \sqrt{6}\xi\phi}, \qquad (12)$$

$B_0 = \mathrm{const}$, while $H(\phi)$ is different for different ξ:

$0 < \xi < 1/6$:

$$H(\phi) = \exp\left[-\frac{\sqrt{1 - 6\xi}}{\sqrt{6\xi}}\arcsin\left(\sqrt{\xi(1 - 6\xi)}\phi\right)\right],$$

$\xi > 1/6$:

$$H(\phi) = \left[\sqrt{\xi(6\xi - 1)}\,\phi + \sqrt{1 + \xi(6\xi - 1)\phi^2}\right]^{\frac{\sqrt{6\xi - 1}}{\sqrt{6\xi}}} \qquad (13)$$

and $H \equiv 1$ for $\xi = 1/6$.

Eq. (11) is valid for $\phi < 1/\sqrt{\xi}$, and the Jordan-frame metric $g_{\mu\nu} = \overline{g}_{\mu\nu}/f$ under the condition (10) can be written in terms of ϕ as follows:

$$ds_{\mathrm{J}}^2 = \frac{BH^2}{1 - \xi\phi^2}\left\{dt^2 - 256h^2\frac{B^2H^4|1 - \xi(1 - 6\xi)\phi^2|}{(1 - \xi\phi^2)^2(1 - B^2H^4)^4}d\phi^2\right.$$

$$\left. - \frac{16h^2d\Omega^2}{(1 - B^2H^4)^2}\right\}. \qquad (14)$$

The metric (14) is extendable to $\phi > 1/\sqrt{\xi}$ since BH^2 vanishes and behaves as $1 - \xi\phi^2$ near $\phi = 1/\sqrt{\xi}$. In this new region, another copy of the solution (6), (7) subject to (10) is valid, but for "primed" quantities,

[4]The prime will designate quantities describing the Einstein frame for $\phi > 1/\sqrt{\xi}$.

[5]We have changed the notations as compared with [6], in particular, we have replaced $\sqrt{6}\Phi_\xi \mapsto \phi$, $H \mapsto 1/H$ and $F^2 \mapsto 1/B$, to avoid imaginary F at $\phi > 1/\sqrt{\xi}$.

where the constants C' and ψ_0' may in principle differ from C and $\psi_0 = 0$. When one constructs a metric similar to (14) from this solution, it will then contain C' instead of C and $B\,\mathrm{e}^{\psi_0'\sqrt{2/3}}$ instead of B. Since the continued Jordan-frame metric should be smooth at $\phi = 1/\sqrt{\xi}$, we conclude that $C' = C$ and $\psi_0' = \psi_0 = 0$. (It actually suffices to require the continuity of g_{tt} and $g_{\phi\phi}$ at $\phi = 1/\sqrt{\xi}$ to reach this conclusion, and the resulting metric turns out to be smooth.)

The coordinate ϕ covers the whole manifold $\mathbb{M}[g]$, and it is now possible to study its behaviour beyond $\mathbb{S}_{\mathrm{trans}}$.

In case $\xi > 1/6$, for any B_0, with growing ϕ the quantity $B^2 H^4$ eventually reaches the value 1, where $g_{\theta\theta} \to \infty$, i.e., we arrive at another spatial asymptotic, and it is straightforward to verify that it is flat. In other words, we obtain a wormhole.

In case $\xi < 1/6$ everything depends on B_0. If

$$B_0 > B_0^{\mathrm{cr}} = \exp\left(\pi\sqrt{\frac{1-6\xi}{6\xi}}\right), \tag{15}$$

the situation is the same as for $\xi > 1/6$, i.e., a wormhole. If $B_0 > B_0^{\mathrm{cr}}$, then, while $g_{\theta\theta}$ is still finite, ϕ reaches the value $1/\sqrt{\xi(1-6\xi)}$, a location of a curvature singularity [6]. So we have a naked singularity instead of a wormhole. Lastly, for $B_0 = B_0^{\mathrm{cr}}$, the maximum value of ϕ is again $1/\sqrt{\xi(1-6\xi)}$, but now it is a non-flat spatial infinity.

The case $\xi = 1/6$ (conformal coupling) is simpler and has been analyzed in Refs. [3, 5, 7]. Depending on an integration constant similar to B_0, one obtains either a wormhole, or a naked singularity, or a black hole with a scalar charge [8, 9], whose instability was proved in Ref. [10].

3. Stability analysis

Consider small (linear) spherically symmetric perturbations of the above wormholes. It is helpful to work separately in each of the two Einstein-frame manifolds $\overline{\mathbb{M}}$ and $\overline{\mathbb{M}}'$, perturbing the metric quantities α, β, γ in (7) and the field ψ, replacing

$$\psi(u) \to \psi(u,t) = \psi(u) + \delta\psi(u,t) \tag{16}$$

and similarly for other quantities; the same is done for their counterparts in $\overline{\mathbb{M}}'$. Due to spherical symmetry, the only dynamical degree of freedom is the scalar field, obeying the equation $\Box\psi = 0$, while other perturbations must be expressed in terms of $\delta\psi$ and its derivatives via the Einstein equations. The perturbed scalar equation has the form

$$\mathrm{e}^{-\gamma+\alpha+2\beta}\ddot{\psi} - \left(\mathrm{e}^{\gamma-\alpha+2\beta}\psi_u\right)_u = 0. \tag{17}$$

where the dot stands for $\partial/\partial t$ and the subscript u for the radial coordinate derivative $\partial/\partial x^1$. One can notice that Eq. (17) decouples from perturbations other than

$\delta\psi$ if one chooses the frame of reference and the coordinates in the perturbed space-time (the gauge for short) so that

$$\delta\alpha = 2\delta\beta + \delta\gamma. \tag{18}$$

The relation $\alpha = 2\beta + \gamma$ thus holds for both the static background written as in (6), (7) and the perturbations. The unperturbed part of Eq. (17) reads $\psi_{uu} = 0$ and is satisfied by (6), while for $\delta\psi$ we obtain the wave equation

$$\mathrm{e}^{4\beta(u)}(\delta\psi)\ddot{} - \psi_{uu} = 0. \tag{19}$$

The static nature of the background solution makes it possible to separate the variables,

$$\delta\psi = \Phi(u)\,\mathrm{e}^{i\omega t}, \tag{20}$$

and to reduce the stability problem to a boundary-value problem for $\psi(u)$. Namely, if there exists a nontrivial solution to (19) with $\omega^2 < 0$, satisfying some physically reasonable boundary conditions, then the static background system is unstable since the perturbations can exponentially grow with t. Otherwise it is stable in the linear approximation.

Suppose $-\omega^2 = \Omega^2$, $\Omega > 0$. The equation that follows directly from (19),

$$\Phi_{uu} - \Omega^2\,\mathrm{e}^{4\beta(u)}\Phi = 0, \tag{21}$$

is converted to the normal Liouville (Schrödinger-like) form

$$\begin{aligned} d^2y/dx^2 - [\Omega^2 + V(x)]y(x) &= 0, \\ V(x) &= \mathrm{e}^{-4\beta}(\beta_{uu} - \beta_u{}^2). \end{aligned} \tag{22}$$

by the transformation

$$\Phi(u) = y(x)\,\mathrm{e}^{-\beta}, \qquad x = -\int \mathrm{e}^{2\beta(u)}du. \tag{23}$$

Eq. (22) makes it possible to use the experience of quantum mechanics (QM): Ω^2 here corresponds to $-E$ in the Schrödinger equation. In other words, the presence of "negative energy levels" $E = -\Omega^2 < 0$ for the potential $V(x)$ indicates the instability of our system.

$V(x)$ is written explicitly as a function of u:

$$V(u) = -\frac{\sinh^3(2hu)}{32h^2}\left(\mathrm{e}^{-2hu} - 9\,\mathrm{e}^{-6hu}\right). \tag{24}$$

The variable x behaves as follows at small and large u:

$u \to 0$ (spatial infinity): $x \approx \mathrm{e}^{\beta} \approx 1/u$;

$u \to \infty$ (the sphere $\mathbb{S}_{\mathrm{trans}}$): $x \approx 8h\,\mathrm{e}^{-2hu}$.

For the potential $V(x)$ one finds:

$$\begin{aligned} V(x) &\approx 2h/x^3 \quad (x \to \infty \quad \text{— spatial asymptotic}), \\ V(x) &\approx -1/(4x^2)\ (x \to 0 \text{ — the sphere } \mathbb{S}_{\mathrm{trans}}). \end{aligned} \tag{25}$$

Thus we have a quadratic potential well at $\mathbb{S}_{\mathrm{trans}}$, which is placed at $x = 0$ by choosing the proper value of the arbitrary constant in the definition of x in Eq. (23).

The same form of Eq. (22) with the same potential V is obtained for the Einstein frame $\overline{\mathbb{M}}'$, the other part of the wormhole, due to equal values of the integration constants in $\overline{\mathbb{M}}$ and $\overline{\mathbb{M}}'$. It makes sense, however, to change $x \to -x$, which does not affect Eq. (22) but makes it possible to unify the perturbation equations for the two parts of $\mathbb{M}$, the space-time of the Jordan-frame. We thus have Eq. (22) with $x \in \mathbb{R}$ and an even function $V(x) = V(-x)$, providing a potential well of the form $V \approx 1/(4x^2)$ near $x = 0$.

The boundary conditions at both spatial asymptotics are obtained from the requirement that the perturbations should possess finite energy. This requirement upon the perturbed EMT leads to the condition $xy \to 0$ as $x \to \pm\infty$. Meanwhile, the asymptotic form of any solution of (22) with $\Omega > 0$ at large $|x|$ is

$$y \approx C_1 \, \mathrm{e}^{\Omega|x|} + C_2 \, \mathrm{e}^{-\Omega|x|}, \qquad C_{1,2} = \text{const.} \qquad (26)$$

Therefore an admissible solution is the one with $C_1 = 0$, with only a decaying exponential. Actually, the conditions at both infinities are that $y \to 0$, i.e., coincide with the boundary conditions for the one-dimensional wave function under the same potential in QM.

As is evident from QM (see, e.g., [11]), a potential well of the form $V \approx 1/(4x^2)$ always possesses negative energy levels, $E = -\Omega^2 < 0$; moreover, the absolute value of Ω has no upper bound. The latter statement can be proved, e.g., by comparing Eq. (22) with our $V(x)$ and with rectangular potentials $\tilde{V} \geq V$ for which $y(x)$ and E are easily found; one can then use the fact that $E_{\min}[V] < E_{\min}[\tilde{V}]$ where $E_{\min}$ is the lowest energy level (ground state) for a given potential.

Recalling that Ω is the perturbation growth increment, we can conclude that our wormholes decay instantaneously within linear perturbation theory. Nonperturbative analysis would probably smooth out this infinite decay rate.

It is of interest to note that at small x the scalar field perturbation $\delta\phi$ behaves as $y\sqrt{|x|}$, hence the smoothness of the "wave function" $y(x)$ at $x = 0$ implies $\delta\phi(0) = 0$. In other words, the perturbation rapidly grows around $\mathbb{S}_{\text{trans}}$ but vanishes on this sphere itself.

4. Concluding remarks

Our perturbation analysis proves the instability of the wormhole solutions for any $\xi > 0$. This violent instability occurs due to a negative pole of the perturbational effective potential $V(x)$ at the sphere $\mathbb{S}_{\text{trans}}$ where vanishes the original function $f(\phi)$ in the action (1), which actually means that the effective gravitational constant, proportional to f^{-1}, blows up and changes its sign.

The instability of black holes with a conformal scalar field, found long ago in Ref. [10], is another example of such a phenomenon. It is quite plausible that instabilities of this kind are a common feature of STT solutions with conformal continuations, such that the transformation (3) maps the Einstein-frame manifold $\overline{\mathbb{M}}[\overline{g}]$ to only a part of the whole Jordan-frame manifold $\mathbb{M}[g]$ and after which the effective gravitational coupling becomes negative. A similar instability was pointed out by Starobinsky [12] for cosmological models with conformally coupled scalar fields. We hope to return to this subject in our future work.

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